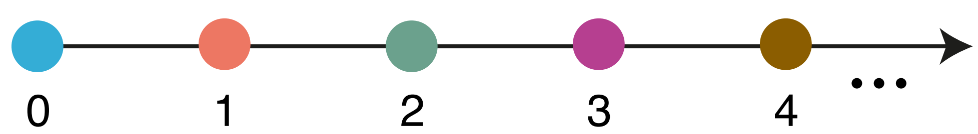
4D Topological Physics with Synthetic Dimensions

Hannah Price
University of Birmingham, UK

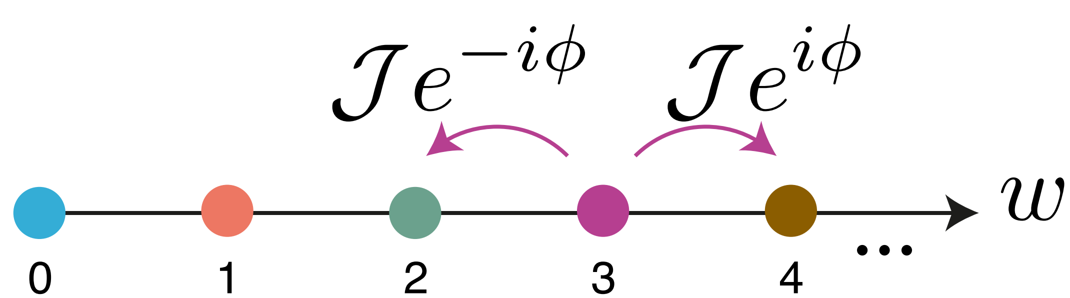
Synthetic Dimensions

General Concept:

1. Identify a set of states and reinterpret as sites in a synthetic dimension



2. Couple these modes to simulate a tight-binding "hopping"



WHY?

- Implement artificial gauge fields
- Reach higher-dimensional models

Boada et al., PRL, 108, 133001 (2012),
Celi et al., PRL, 112, 043001 (2014)

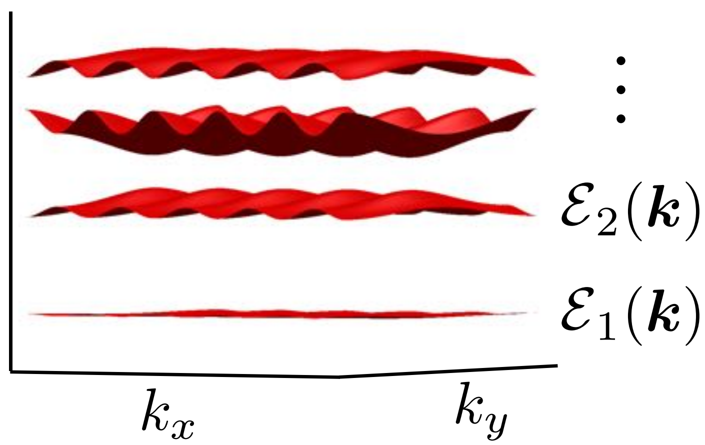
Outline

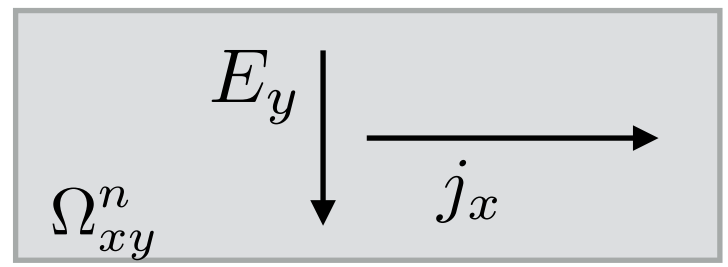
1. **Reminder of the 2D Quantum Hall Effect**

2. 4D Topological Physics

3. 4D Quantum Hall in Synthetic Dimensions

2D Quantum Hall Effect





$$j_x = -\frac{q^2}{h} E_y \sum_{n \in occ.} \nu_1^n$$

band insulator

Geometrical Berry curvature

Bloch states

$\psi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,k}(\mathbf{r})$

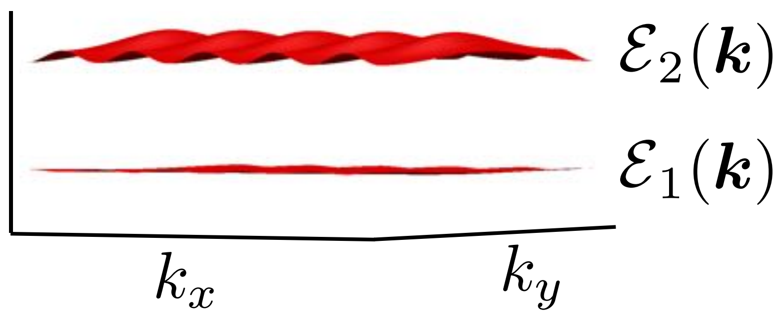
$$\Omega_{xy}^n = i \left[\langle \frac{\partial u_n}{\partial k_x} | \frac{\partial u_n}{\partial k_y} \rangle - \langle \frac{\partial u_n}{\partial k_y} | \frac{\partial u_n}{\partial k_x} \rangle \right]$$

Topological First Chern number

$$\nu_1^n = \frac{1}{2\pi} \int_{\mathrm{BZ}} \Omega_{xy}^n \mathrm{d}k_x \mathrm{d}k_y$$

Topological transitions only
when band-gap closes

How to get a 2D QH system?

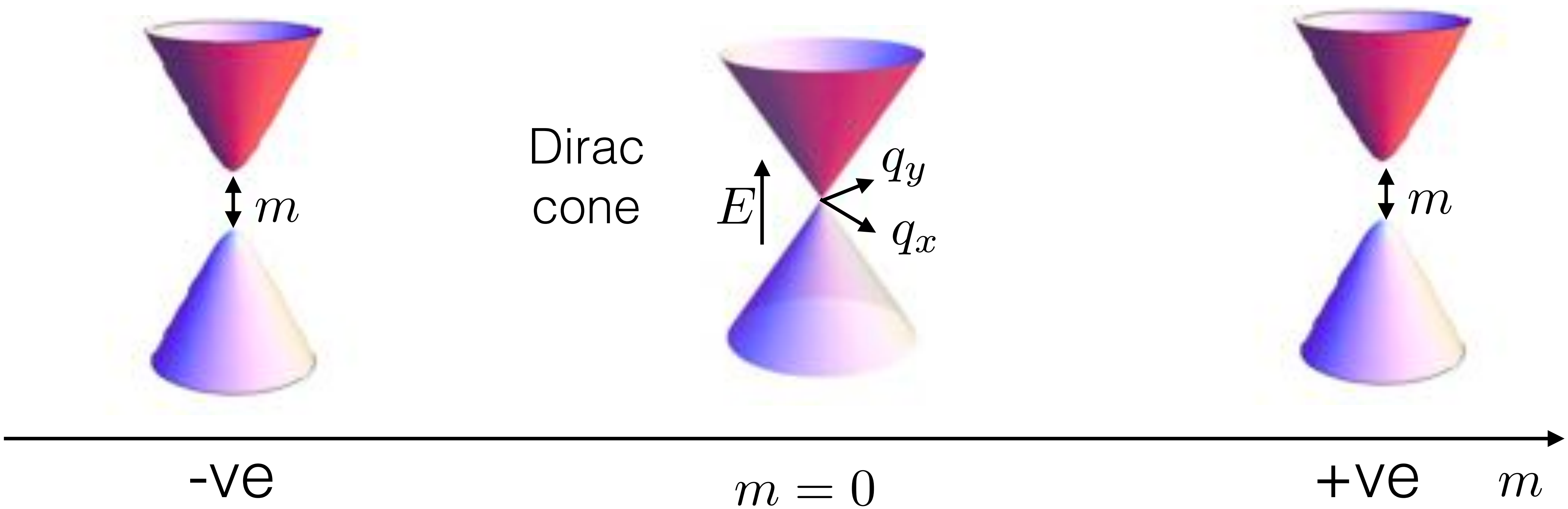


Minimal two-band model, e.g. spinless atoms on lattice with two-site unit cell:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\hat{I} + \mathbf{d}(\mathbf{k}) \cdot \sigma$$

Topological transitions: e.g. at Dirac points

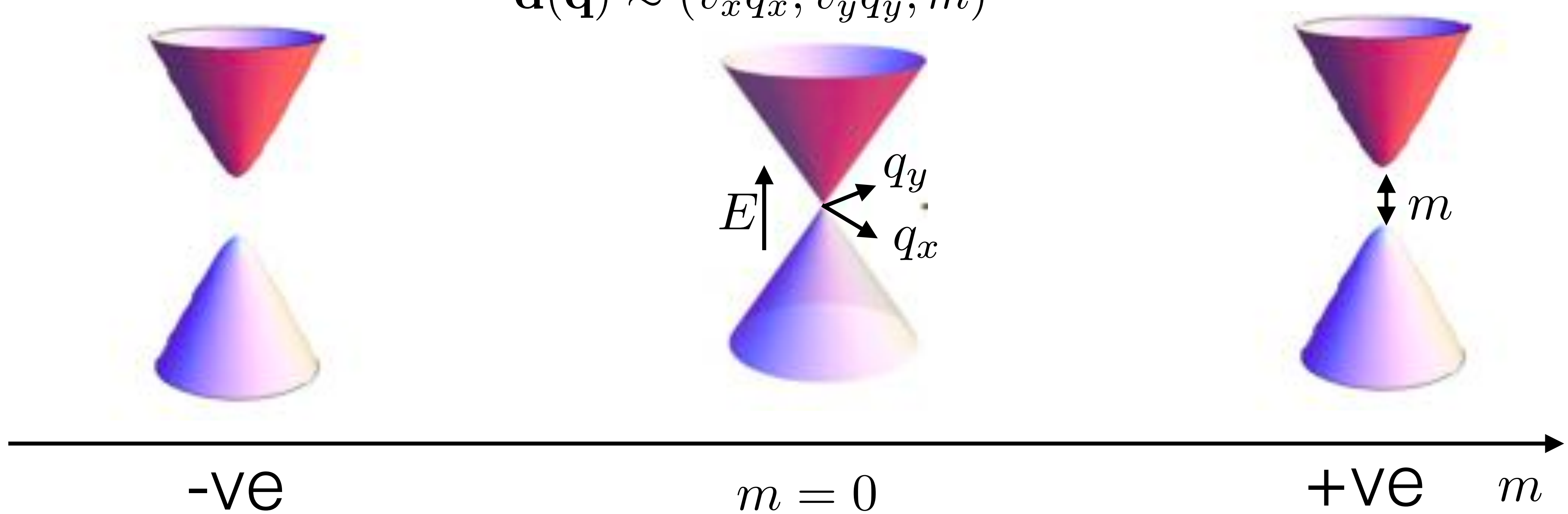
$$H(\mathbf{q}) \approx v_x q_x \sigma_x + v_y q_y \sigma_y + m\sigma_z$$



Berry curvature

$$H(\mathbf{q}) \approx \mathbf{d}(\mathbf{q}) \cdot \sigma \quad \longrightarrow \quad \Omega^{-}_{xy} = \frac{1}{2}\epsilon^{abc}\hat{d}_a \partial_{q_x} \hat{d}_b \partial_{q_y} \hat{d}_c$$

$$\mathbf{d}(\mathbf{q}) \approx (v_x q_x, v_y q_y, m)$$



Berry curvature flips across transition as $d_3 = -m \rightarrow d_3 = m$

Type 1*:** d_1, d_2 same signs —> ***increases Ω^{-}_{xy}

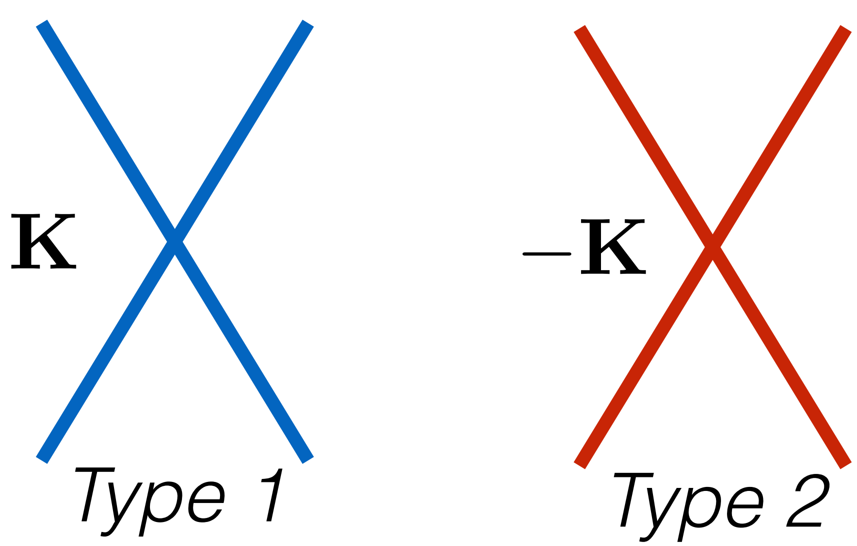
Type 2*:** d_1, d_2 opposite signs —> ***decreases Ω^{-}_{xy}

Chern Number

Time-reversal symmetry for **spinless** particles

$$H^*(\mathbf{k}) = H(-\mathbf{k})$$ implies $$d_{1,3}(\mathbf{k}) = d_{1,3}(-\mathbf{k})$$ $$d_2(\mathbf{k}) = -d_2(-\mathbf{k})$$ as $\sigma_y^* = -\sigma_y$

So have **TRS pairs** of **opposite type**



$$\nu_1^- = \frac{1}{4\pi}\int_{\mathrm{BZ}} \Omega_{xy}^- \mathrm{d}k_x \mathrm{d}k_y$$

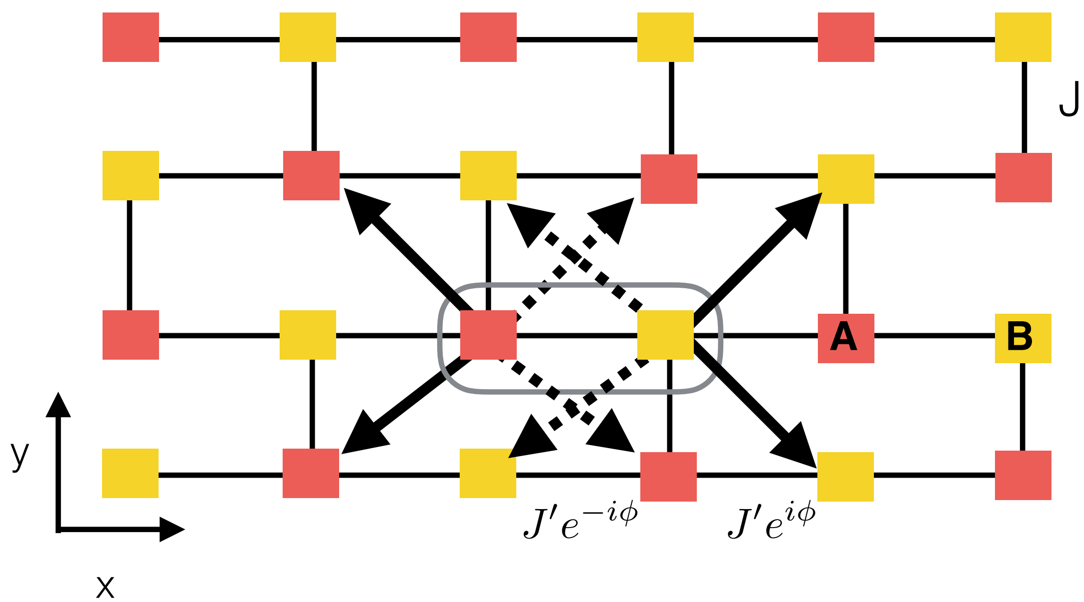
transitions are topologically trivial with TRS

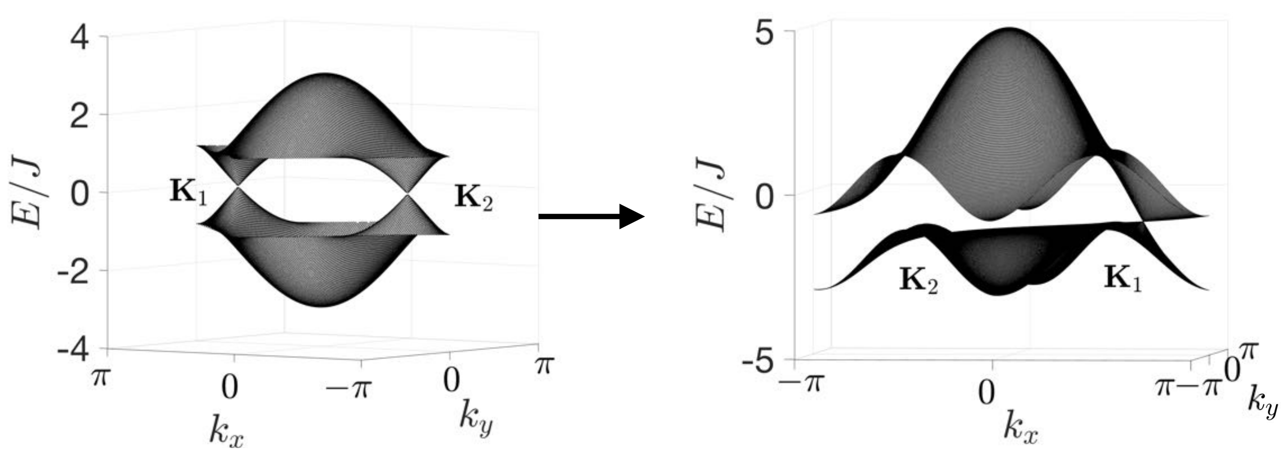
Breaking Time-Reversal Symmetry

e.g.

Haldane model:

Haldane, PRL 61, 2015 (1988)

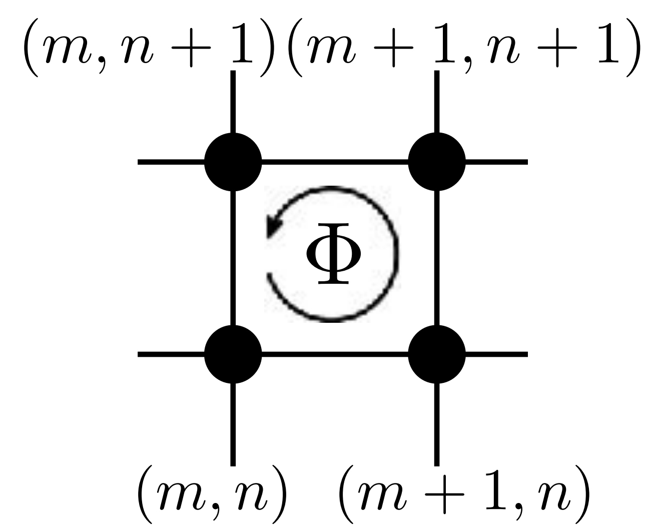


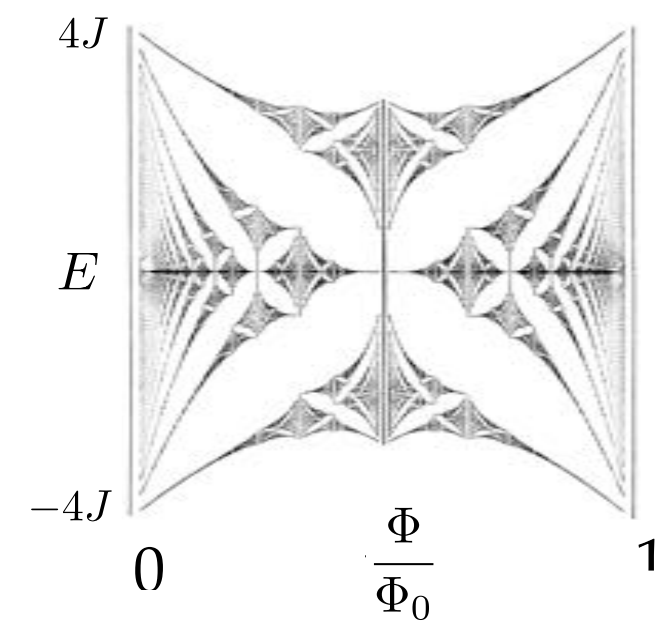


Cold atom experiments:
Jotzu et al, Nature 515, 237 (2014)
Flaschner et al, Nat. Phys. 14, 265 (2018) ….

Landau levels / Hofstadter model:

Hofstadter, PRB, 14, 2239, 1976





$$\mathcal{H} = J\sum_{m,n}(\hat{c}^{\dagger}_{m+1,n}\hat{c}_{m,n} + e^{i2\pi\Phi m}\hat{c}^{\dagger}_{m,n+1}\hat{c}_{m,n}) + \text{h.c.}$$

Cold atom experiments:
Aidelsburger et al., PRL, 111, 185301 (2013),
Miyake et al, PRL, 111, 185302 (2013),
Aidelsburger et al., Nat. Phys, 11,162. (2015)….

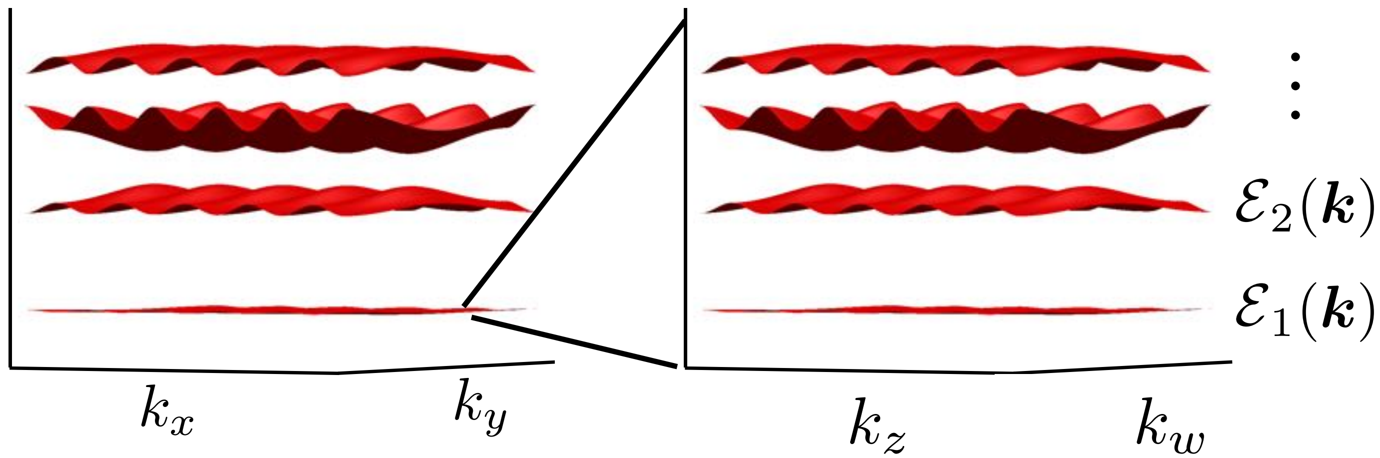
Outline

1. Reminder of the 2D Quantum Hall Effect

2. **4D Topological Physics**

3. 4D Quantum Hall in Synthetic Dimensions

Second Chern Number



$$\Omega = \frac{1}{2}\Omega^{\mu\nu}(\mathbf{k})\mathrm{d}\mathbf{k}_\mu \wedge \mathrm{d}\mathbf{k}_\nu$$

$$\Omega_n^{\mu\nu} = i\left[\langle\frac{\partial u_n}{\partial k_\mu}|\frac{\partial u_n}{\partial k_\nu}\rangle - \langle\frac{\partial u_n}{\partial k_\nu}|\frac{\partial u_n}{\partial k_\mu}\rangle\right]$$

First Chern number

$$\nu_1 = \frac{1}{2\pi}\int_{2\mathrm{DBZ}} \Omega = \frac{1}{2\pi}\int_{2\mathrm{DBZ}} \Omega^{xy}\mathrm{d}k_x\mathrm{d}k_y$$

Second Chern number

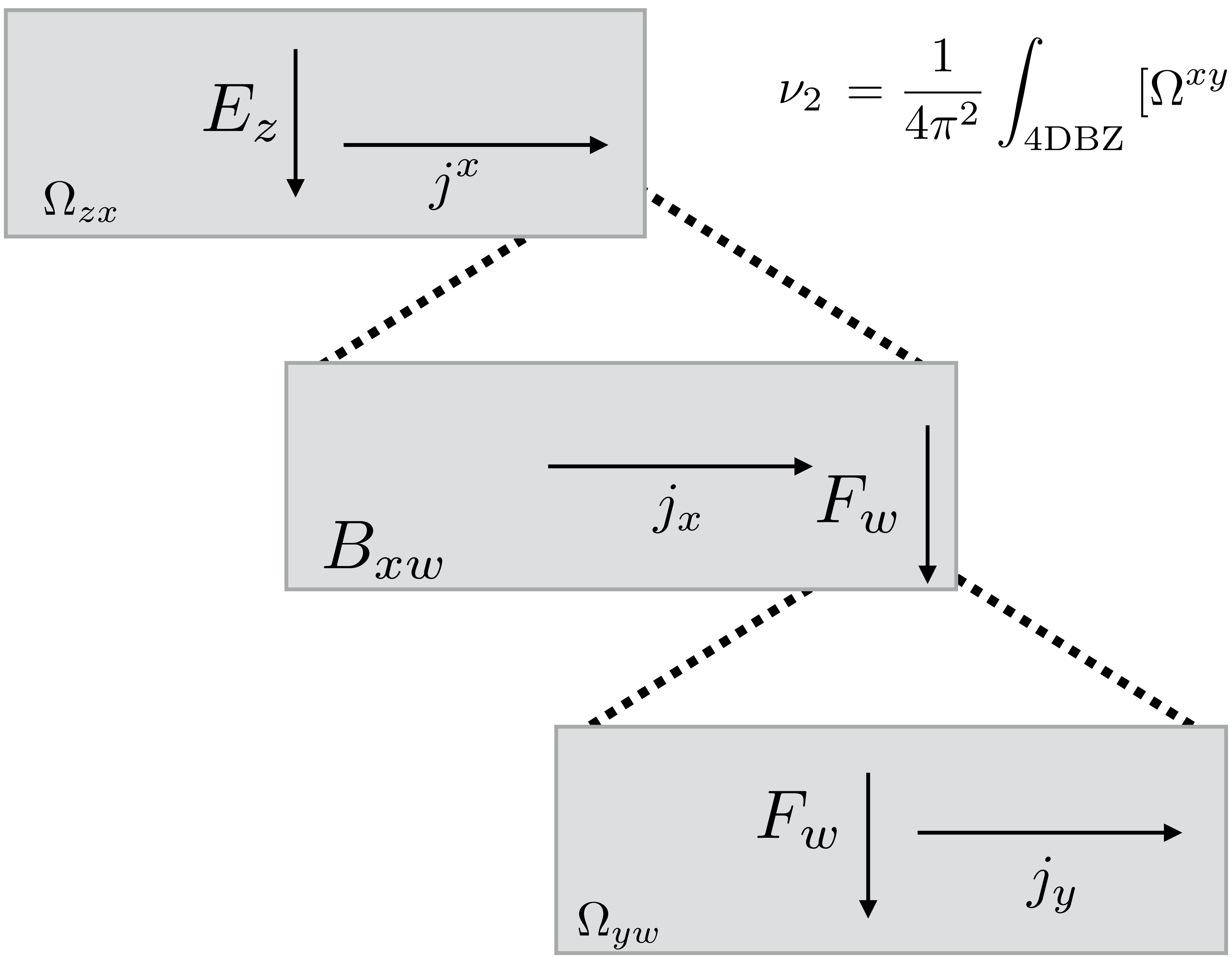
$$\nu_2 = \frac{1}{8\pi^2}\int_{4\mathrm{DBZ}} \Omega\wedge\Omega \in \mathbb{Z}$$
$$= \frac{1}{4\pi^2}\int_{4\mathrm{DBZ}} \left[\Omega^{xy}\Omega^{zw} + \Omega^{wx}\Omega^{zy} + \Omega^{zx}\Omega^{yw}\right]\mathrm{d}^4k$$

(and then the third Chern number in 6D…)

for 6DQH see Petrides, HMP, Zilberberg arXiv:1804.01871 and references there-in

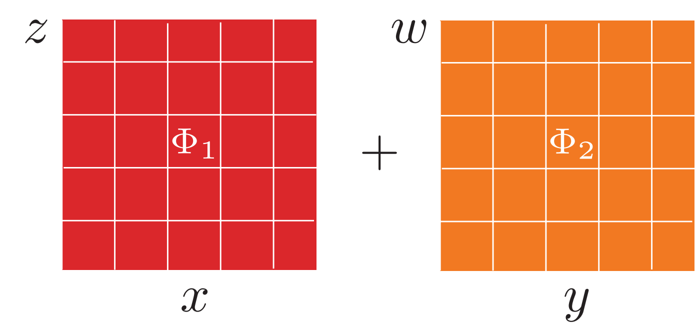
4D Quantum Hall Effect

Very simplest example: 4D Harper-Hofstadter Model



$$\nu_2 = \frac{1}{4\pi^2}\int_{4DBZ} [\Omega^{xy}\Omega^{zw} + \Omega^{wx}\Omega^{zy} + \Omega^{zx}\Omega^{yw}]\, d^4k$$

$$\nu_2 = \nu_1^{zx}\nu_1^{yw}$$



Response to two perturbations:

$$B_{xw} = \partial_x A_w - \partial_w A_x$$
$$E_z$$

$$j_y = -\frac{q^3}{h^2} E_z B_{xw} \nu_2^n$$

Zhang et al, Science 294, 823 (2001)….
HMP, Zilberberg, Ozawa, Carusotto & Goldman, PRL 115, 195303 (2015)
HMP, Zilberberg, Ozawa, Carusotto & Goldman, PRB 93, 245113 (2016)

What do we need for a 4D QH system?

Kitaev, arXiv:0901.2686
Ryu et al., NJP, 12, 2010,
Chiu et al RMP, 88, 035005 (2016)…

	Symmetries			Dimensions							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

1. Preserved TRS for fermions: particles in spin-dependent gauge fields

Zhang et al, Science 294, 823 (2001), Qi et al, Phys. Rev. B 78, 195424 (2008)…..

2. Broken TRS: **4D Harper-Hofstadter model**

Kraus et al, Phys. Rev. Lett. 111, 226401 (2013), HMP et al. 115, 195303 (2015)…

3. Preserved TRS for spinless particles: just lattice connectivity!

HMP, arXiv:1806.05263

What do we need for a 4D QH system?

Kitaev, arXiv:0901.2686
Ryu et al., NJP, 12, 2010,
Chiu et al RMP, 88, 035005 (2016)…

	Symmetries			Dimensions							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

1. Preserved TRS for fermions: particles in spin-dependent gauge fields
   Zhang et al, Science 294, 823 (2001), Qi et al, Phys. Rev. B 78, 195424 (2008)…..

2. Broken TRS: 4D Harper-Hofstadter model
   Kraus et al, Phys. Rev. Lett. 111, 226401 (2013), HMP et al. 115, 195303 (2015)…

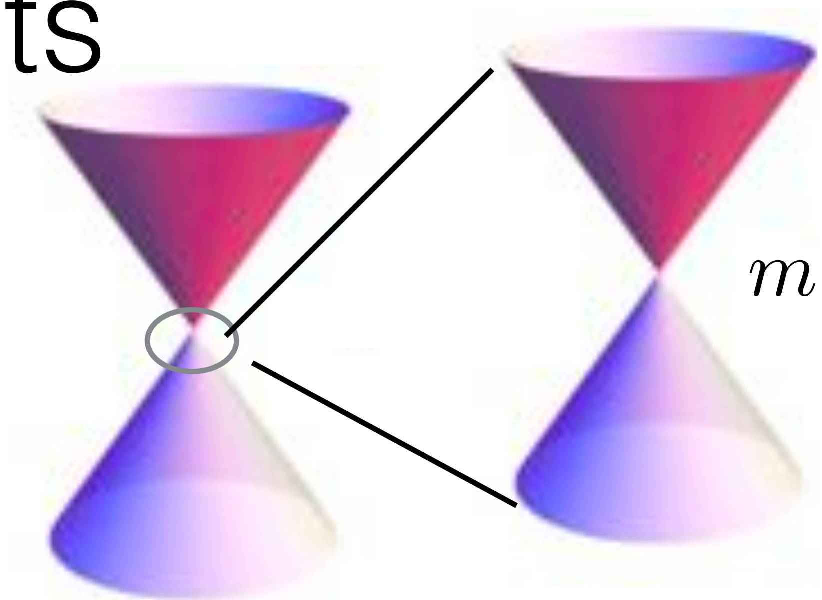
3. **Preserved TRS for spinless particles:** just lattice connectivity!

HMP, arXiv:1806.05263

4D Dirac points

Minimal four-band model:

$$H(\mathbf{k}) = \varepsilon(\mathbf{k})\Gamma_0 + \mathbf{d}(\mathbf{k}) \cdot \mathbf{\Gamma}$$



$$m = 0$$

$$\mathbf{d}(\mathbf{q}) \approx (v_x q_x, v_y q_y, v_z q_z, v_w q_w, m)$$

Qi et al, Phys. Rev. B 78, 195424 (2008)

$$\nu_2^- = \frac{3}{8\pi^2} \int_{\mathrm{BZ}} d^4\mathbf{k} \epsilon^{abcde} \hat{d}_a \partial_{k_x} \hat{d}_b \partial_{k_y} \hat{d}_c \partial_{k_z} \hat{d}_d \partial_{k_w} \hat{d}_e$$

As $d_5 = -m \to d_5 = m$

Type 1: d_1, d_2, d_3, d_4 even no/ minus signs —> ***increases*** integrand

Type 2: d_1, d_2, d_3, d_4 odd no/ minus signs —> ***decreases*** integrand

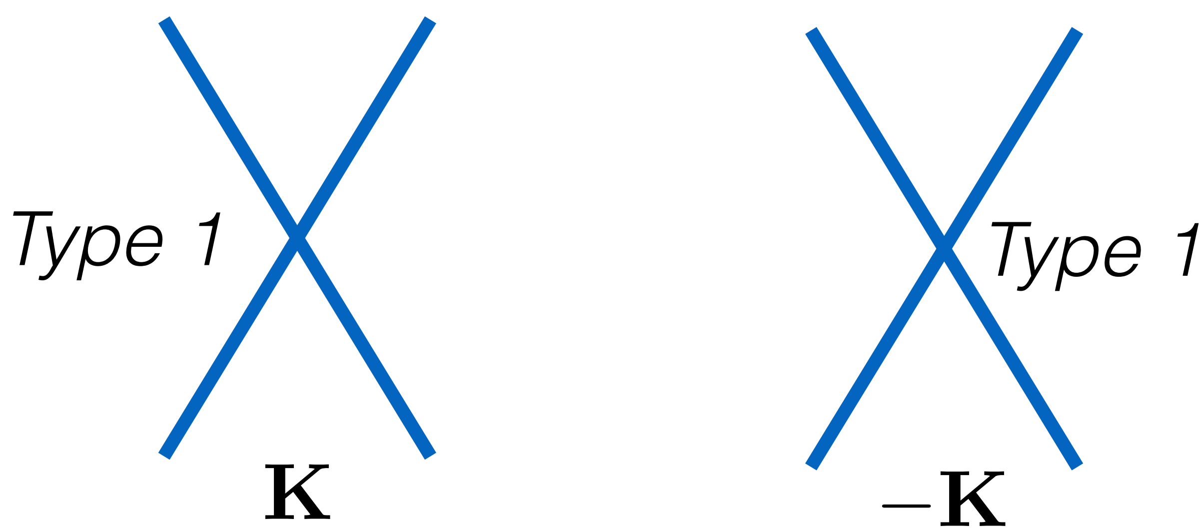
$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

4D Dirac points

Again TRS for spinless particles

$$H^*(\mathbf{k}) = H(-\mathbf{k}) \quad \text{implies} \quad \begin{array}{l} d_{1,3,5}(\mathbf{k}) = d_{1,3,5}(-\mathbf{k}) \\ d_{2,4}(\mathbf{k}) = -d_{2,4}(-\mathbf{k}) \quad \text{as } \Gamma^*_{2,4} = -\Gamma_{2,4} \end{array}$$

So have TRS pairs of *the same* type



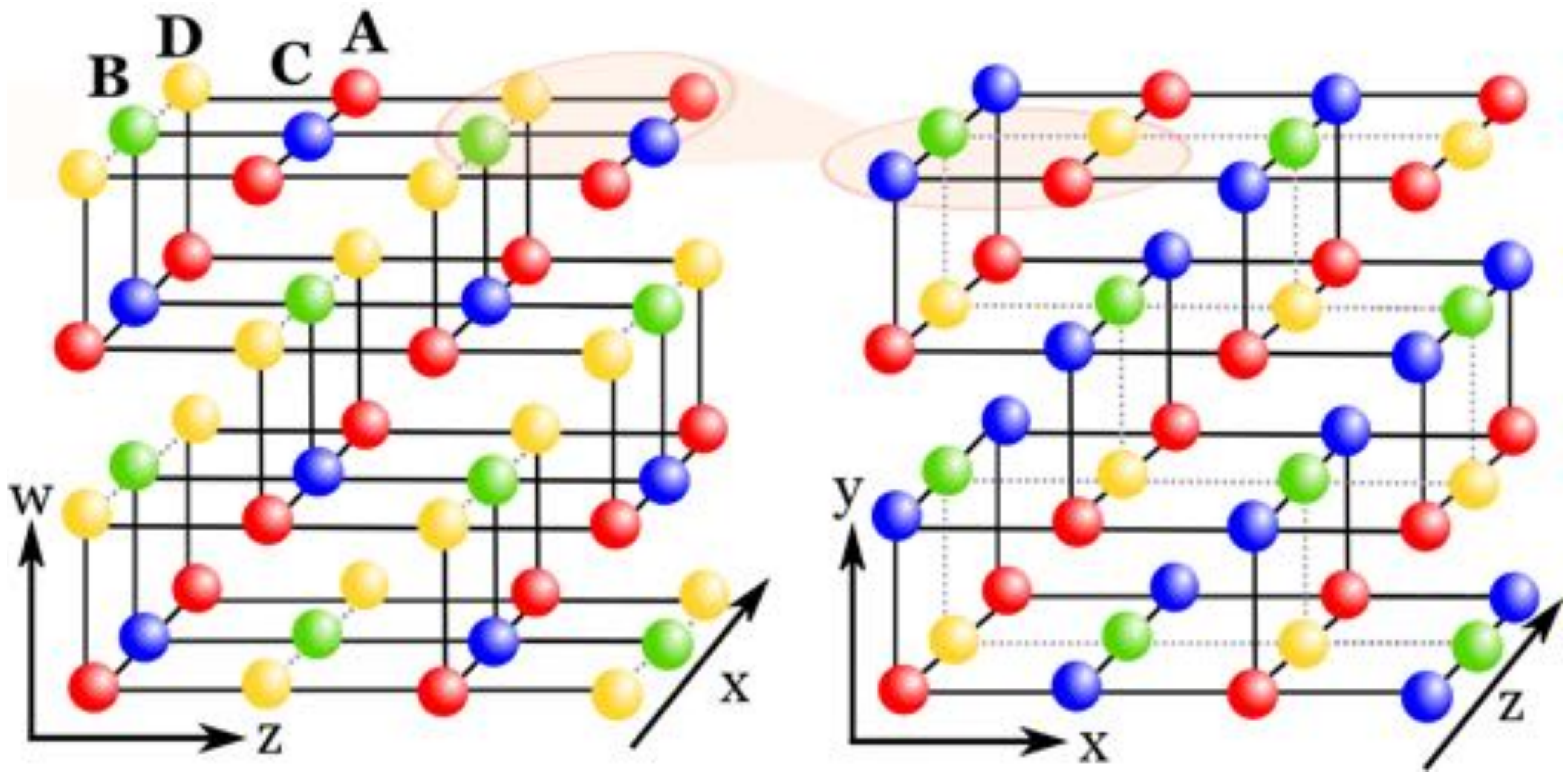
$$\nu_2^- \in 2\mathbb{Z}$$

can have topological transition with TRS!

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

HMP, arXiv:1806.05263

4D Brickwall Lattice

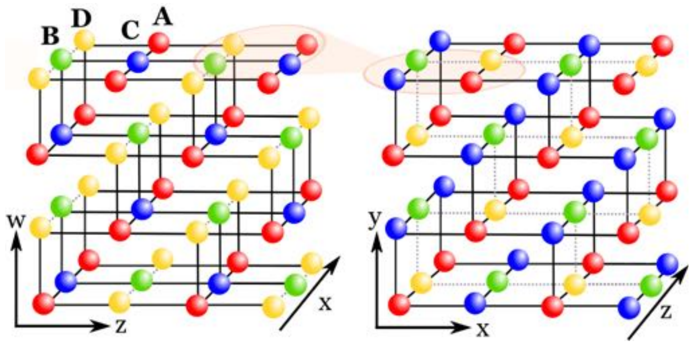


$$H(\mathbf{k}) = J\left[(2\cos k_x + \cos k_y)\Gamma_1 + \sin k_y \Gamma_2 + (2\cos k_z + \cos k_w)\Gamma_3 + \sin k_w \Gamma_4 + m\Gamma_5\right]$$

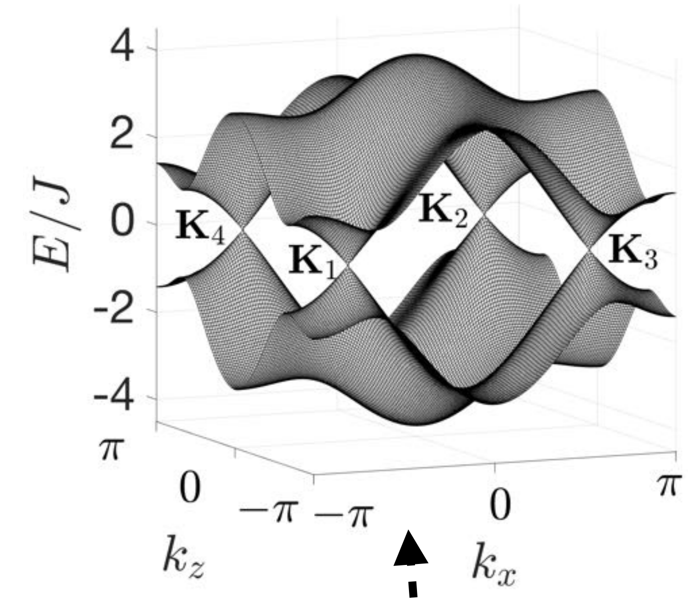
Hopping terms

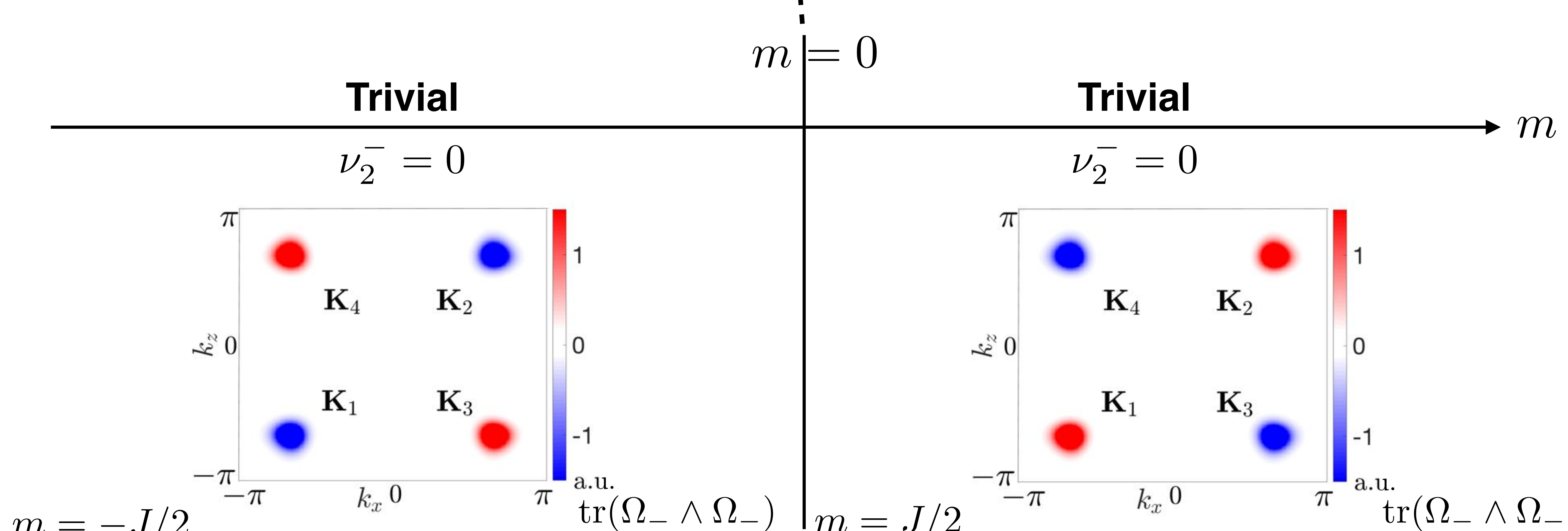
Onsite energies

HMP, arXiv:1806.05263

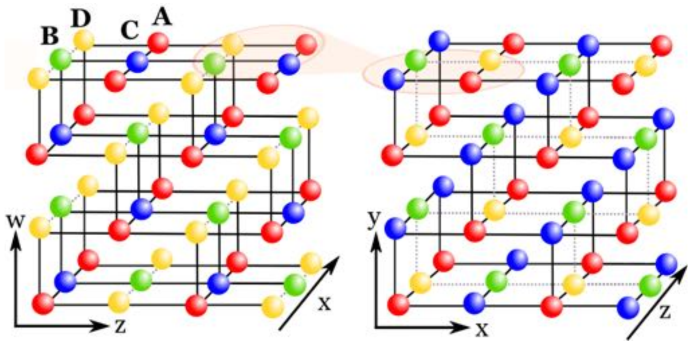


4D Brickwall Lattice

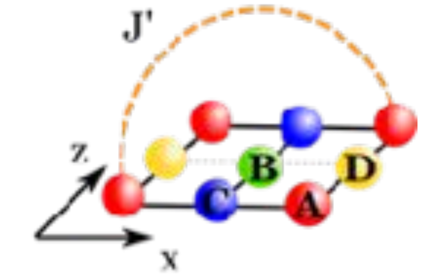




HMP, arXiv:1806.05263

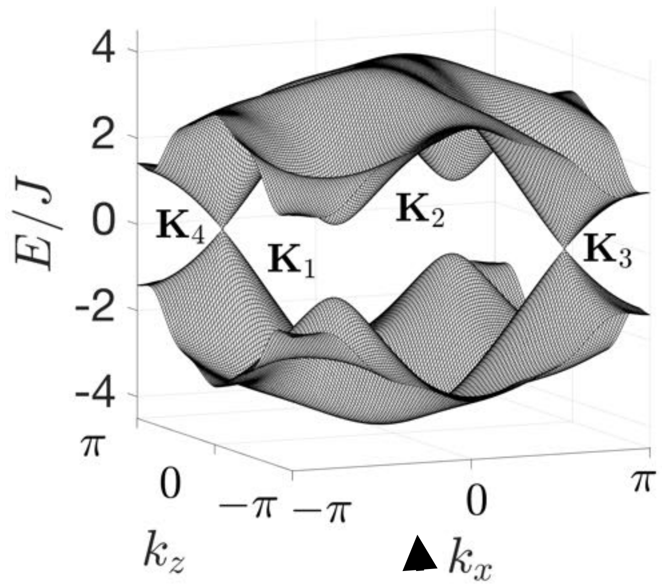


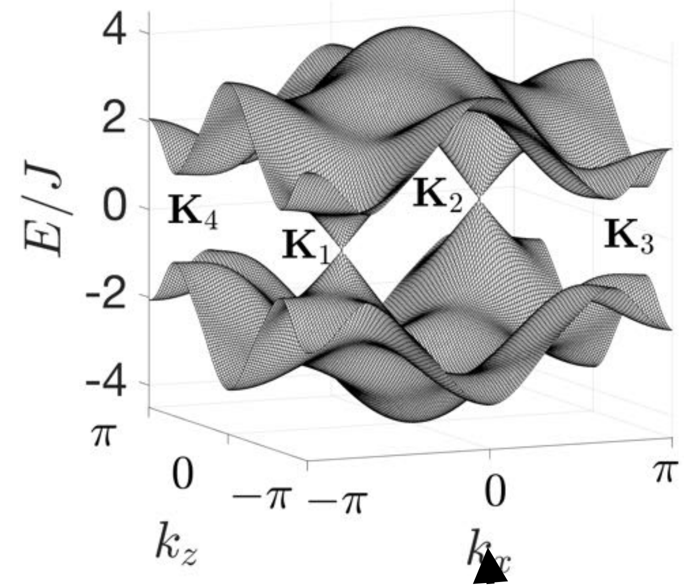
4D Class AI Model

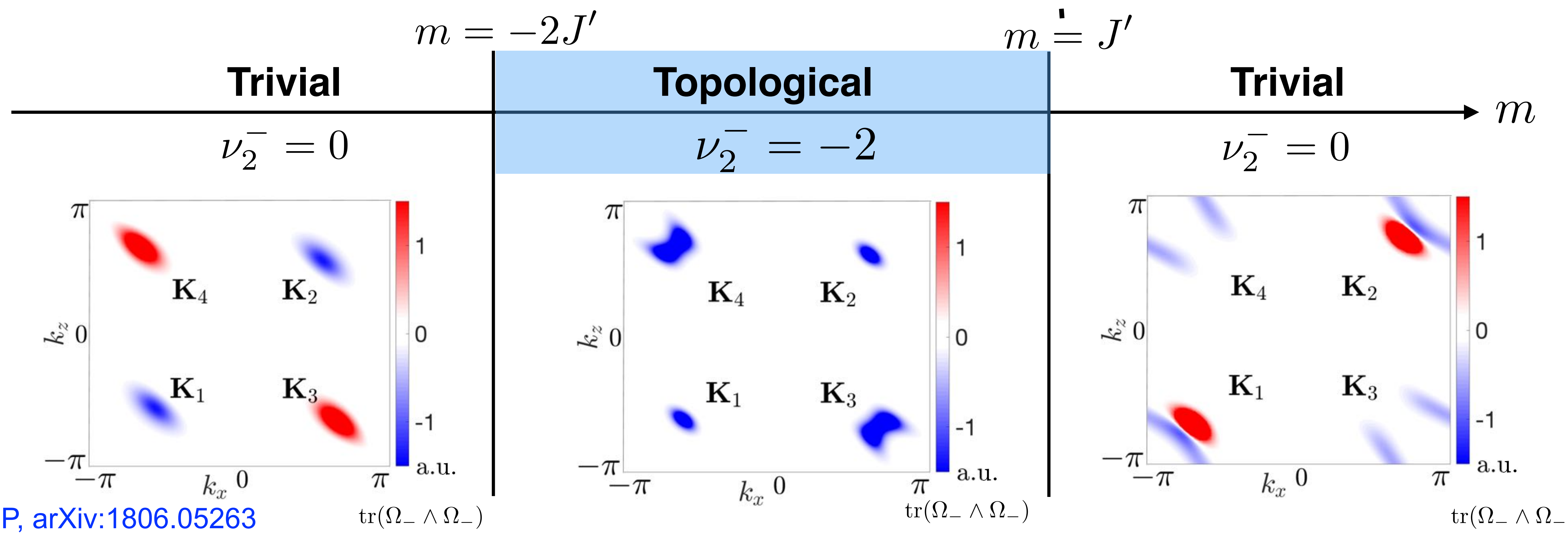


Add real long-range hoppings, e.g.

$$H'(\mathbf{k}) = 2J' \cos(2k_x + 2k_z)\Gamma_5$$







HMP, arXiv:1806.05263

Key points about 4D QH Systems

- Bands labelled by integer **second** Chern numbers

- Quantized **non-linear** response $j_y = -\frac{q^3}{h^2} E_z B_{xw} \nu_2^n$

- **Different classes** of 4D QH systems

	Symmetries			Dimensions							
Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Outline

1. Reminder of the 2D Quantum Hall Effect

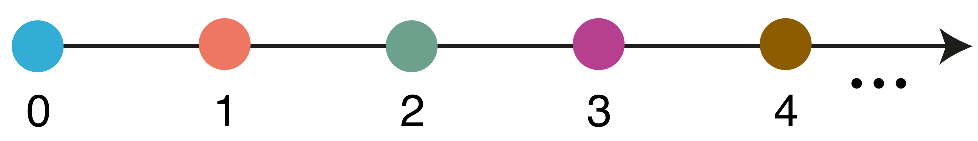
2. 4D Topological Physics

3. 4D Quantum Hall in Synthetic Dimensions

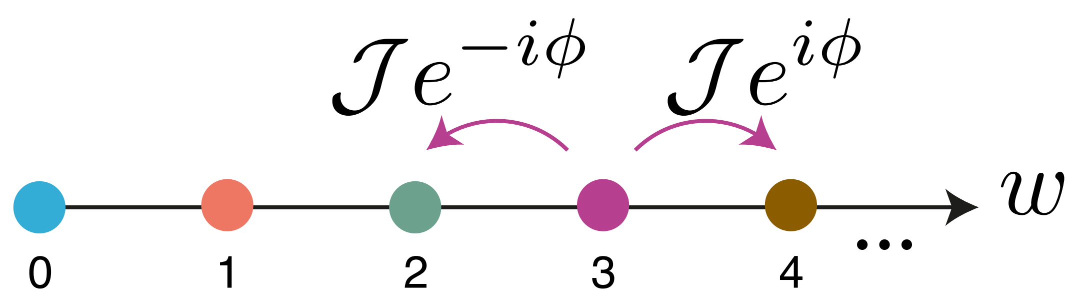
Synthetic Dimensions

General Concept:

1. Identify a set of states and reinterpret as sites in a synthetic dimension



2. Couple these modes to simulate a tight-binding "hopping"



HOW?

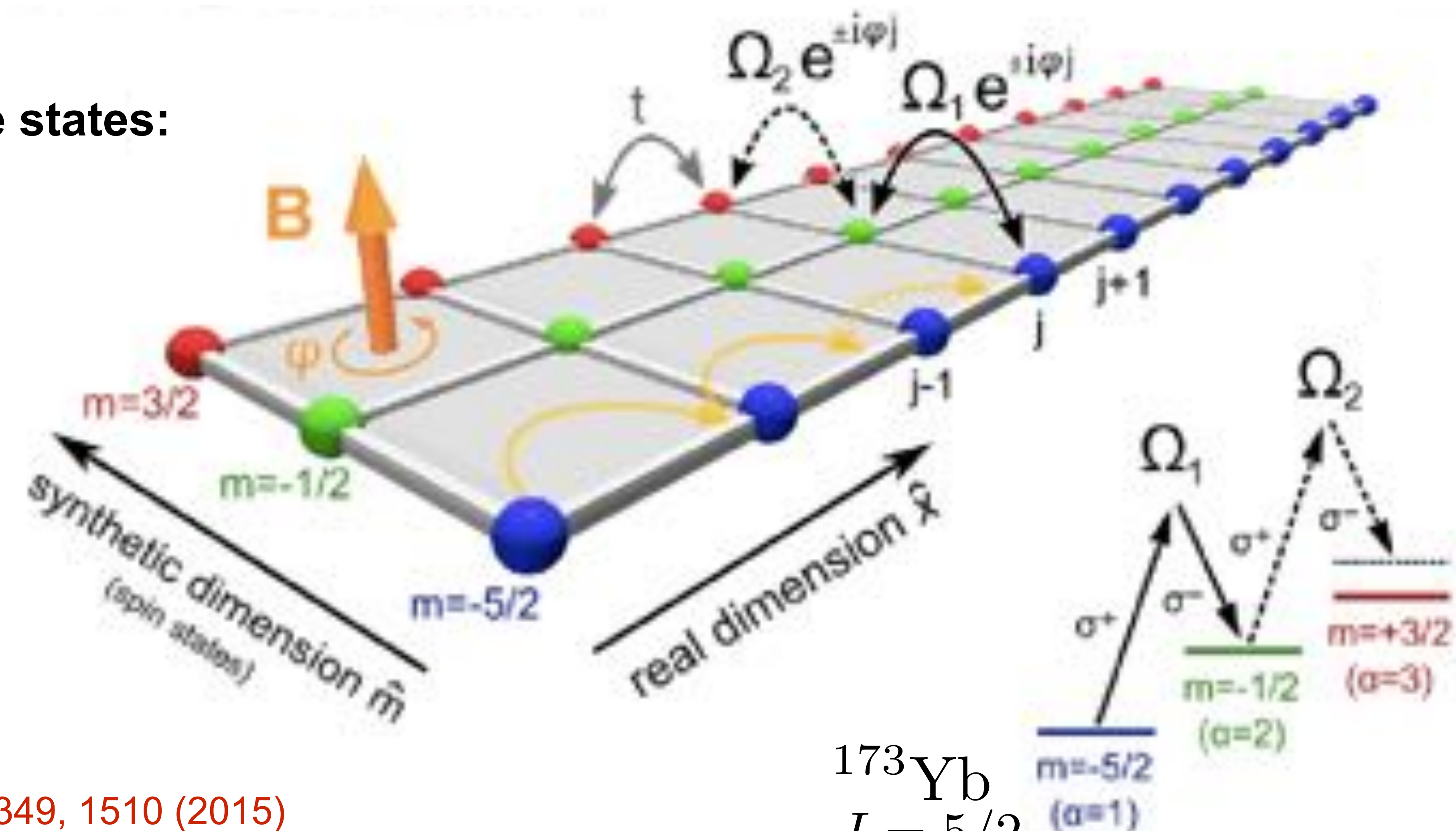
Boada et al., PRL, 108, 133001 (2012),
Celi et al., PRL, 112, 043001 (2014)

Synthetic dimension with internal atomic states

Ingredients:

1. Reinterpret states as sites in synthetic dimension -> **Internal atomic states**
2. Couple states to simulate a "hopping" term -> **Coupling lasers**

For atomic hyperfine states:



$^{173}\mathrm{Yb}$
$I = 5/2$

Florence: Mancini et al, Science, 349, 1510 (2015)
Maryland: Stuhl et al. Science, 349, 1514 (2015)

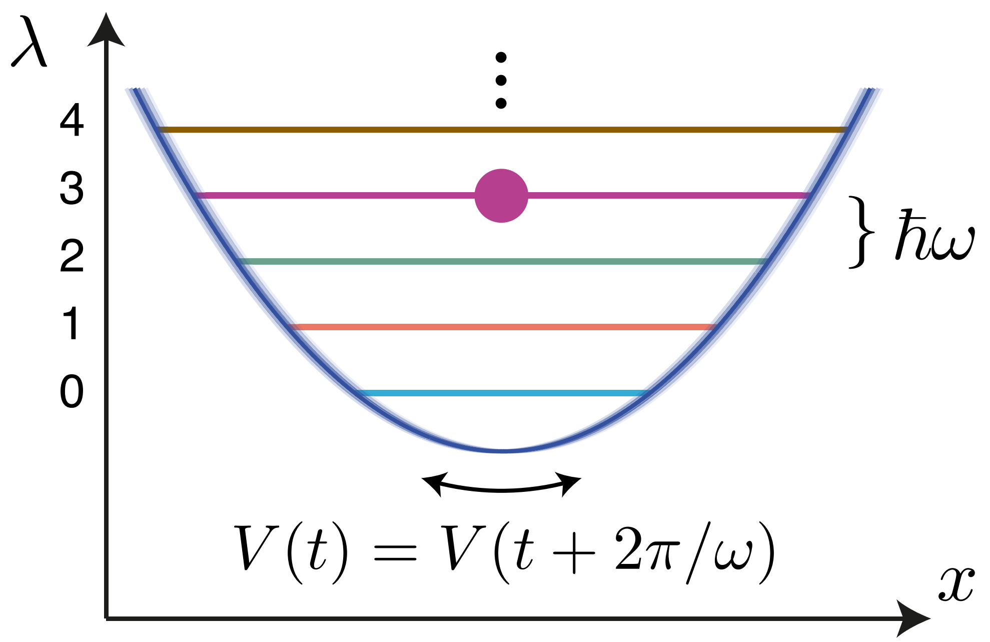
Also now with clock transitions:

Florence Livi et al, Phys. Rev. Lett. 117, 220401 (2016)
Boulder: Kolkowitz et al, Nature, 542, 66 (2017)

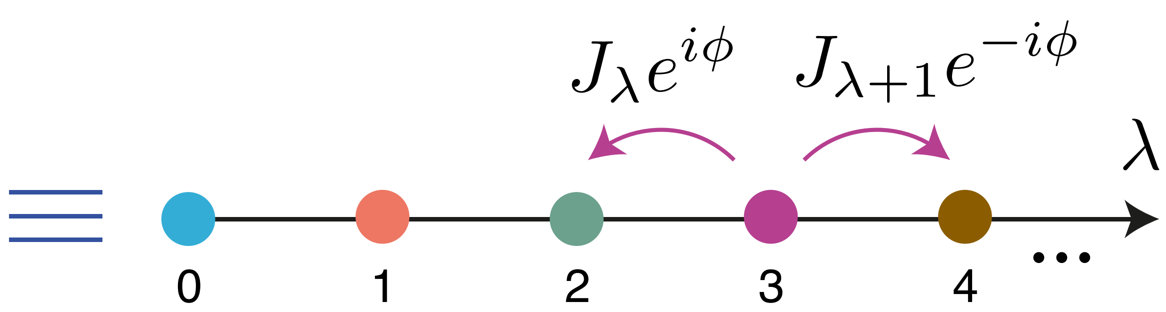
Synthetic dimension with harmonic trap states

Ingredients:

1. Reinterpret states as sites in synthetic dimension -> **Harmonic oscillator states**
2. Couple states to simulate a "hopping" term -> **Shaking of harmonic trap**



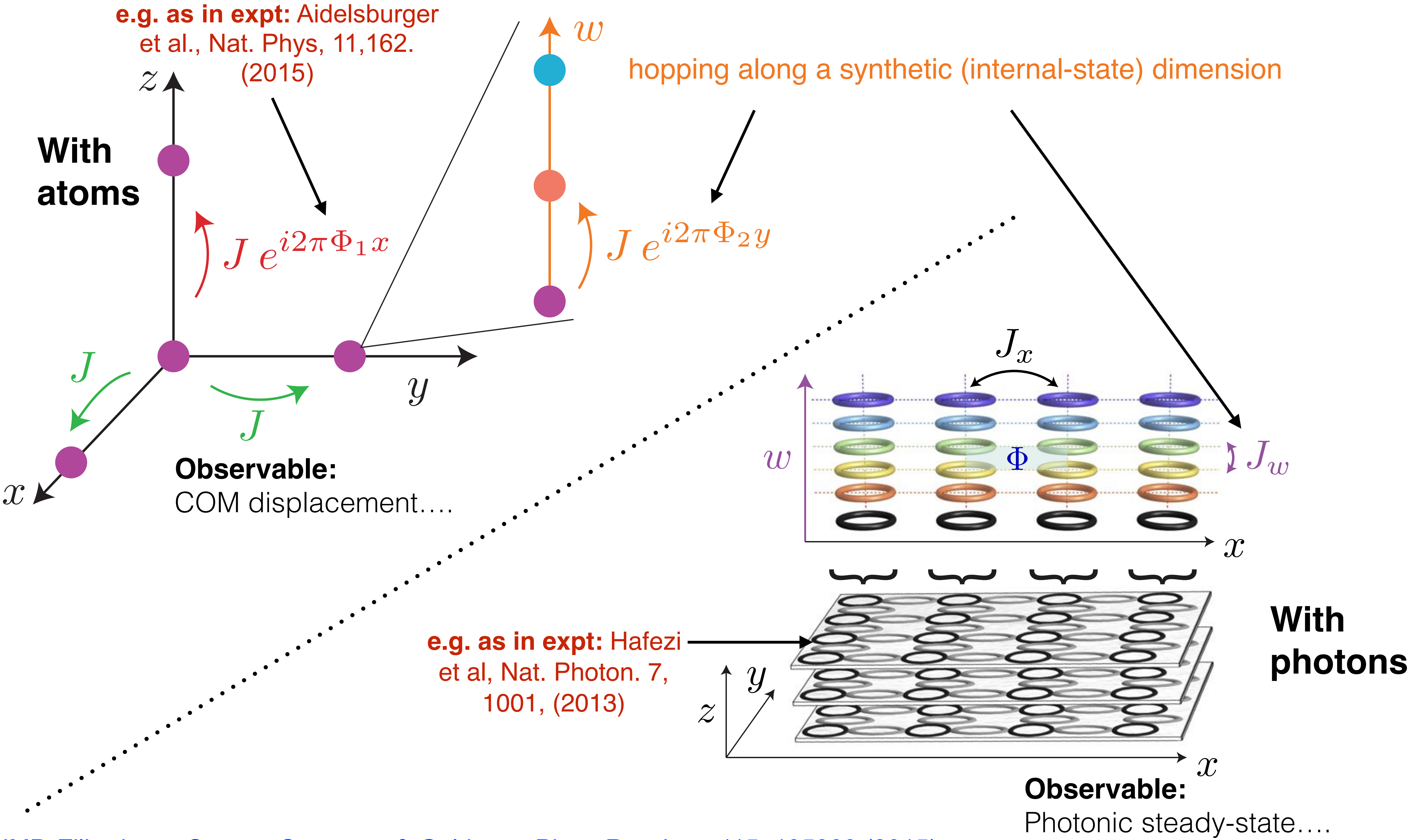
Theory: **HMP**, T. Ozawa and N. Goldman, Phys. Rev. A 95, 023607 (2017)



See poster by Grazia Salerno!

Also: synthetic dimensions for photons:
Optomechanics: Schmidt et al, Optica 2, 7, 635 (2015)
Optical cavities: Luo et al, Nature Comm. 6, 7704, (2015)
Integrated photonics: Ozawa, **HMP**, Goldman, Zilberberg, & Carusotto, Phys. Rev. A 93, 043827 (2016),
L. Yuan, Y. Shi & S. Fan, Optics Letters 41, 4, 741 (2016)
Ozawa & Carusotto, PRL, 118, 013601 (2017)
Waveguides: Lustig et al, arXiv:1807.01983

4D QH with Synthetic Dimensions

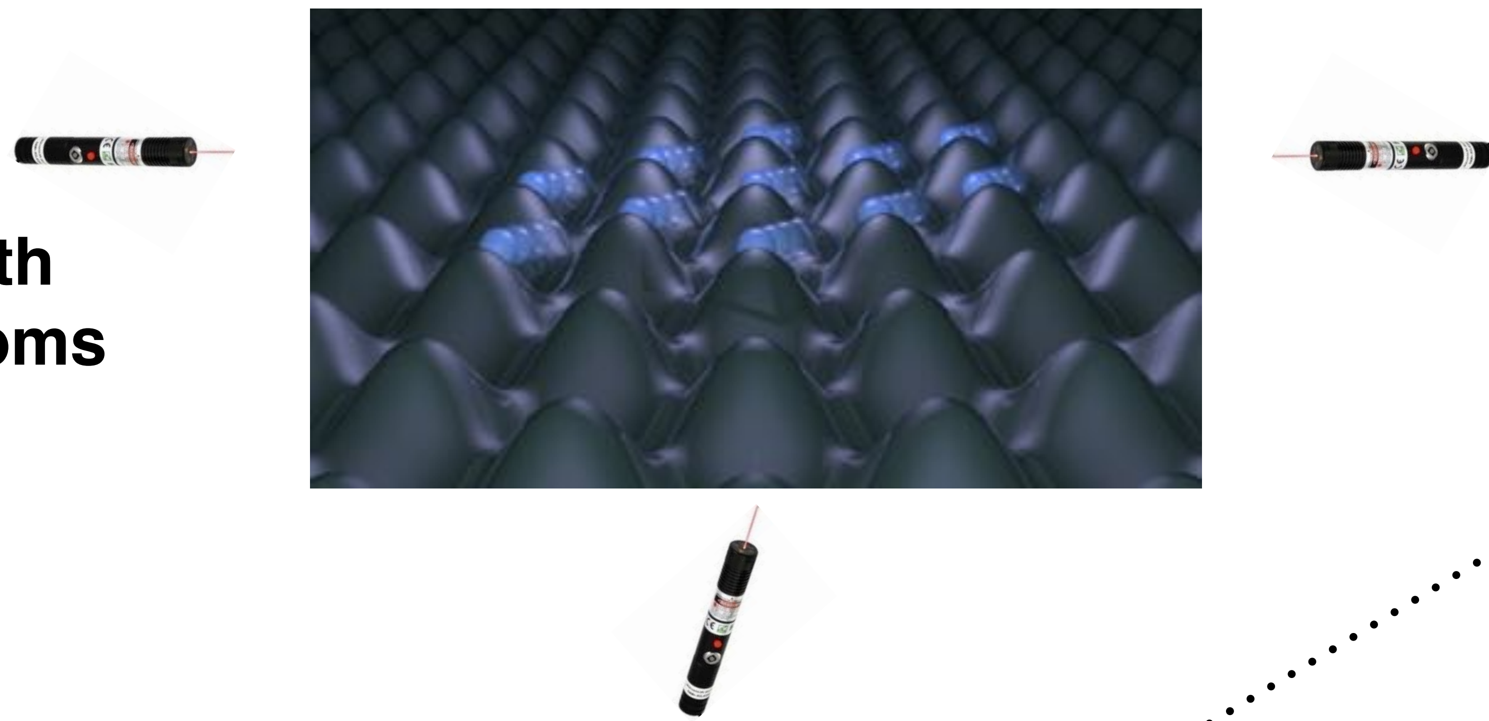


HMP, Zilberberg, Ozawa, Carusotto & Goldman, Phys. Rev. Lett. 115, 195303 (2015)

T. Ozawa, HMP, N. Goldman, O. Zilberberg, and I. Carusotto, Phys. Rev. A 93, 043827 (2016)

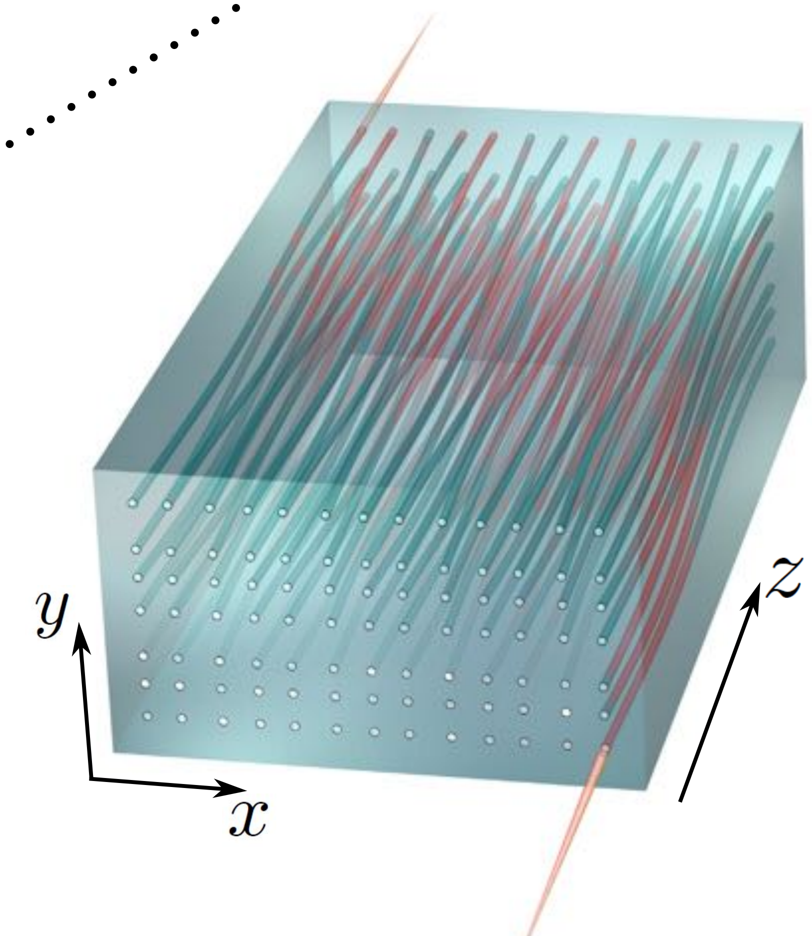
4D QH with Topological Pumping

With atoms



See talk by Oded Zilberberg!

Lohse, Schweizer, HMP, Zilberberg, Bloch, Nature 553, 55–58 (2018)



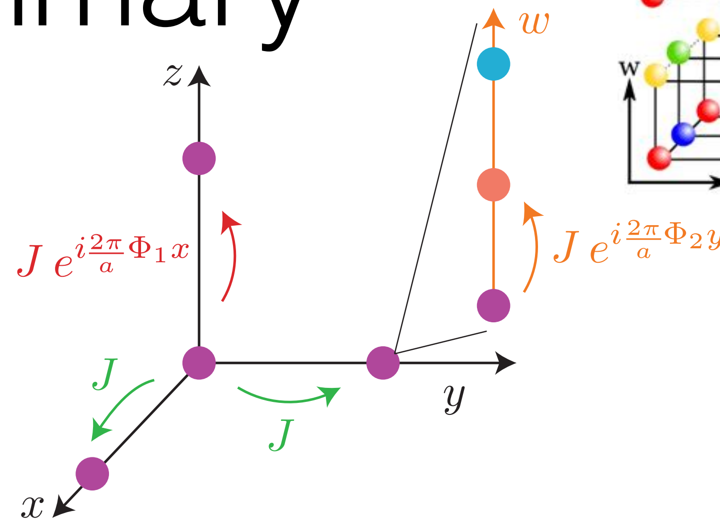
With photons

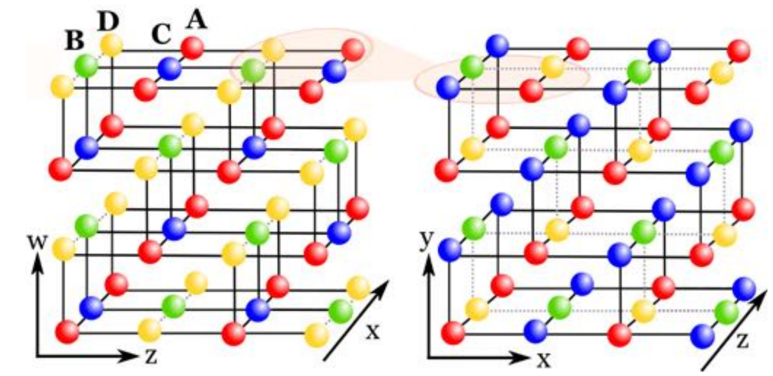
Zilberberg, Huang, Guglielmon, Wang, Chen, Kraus, Rechtsman., Nature 553, 59 (2018)

PhD Position Available!

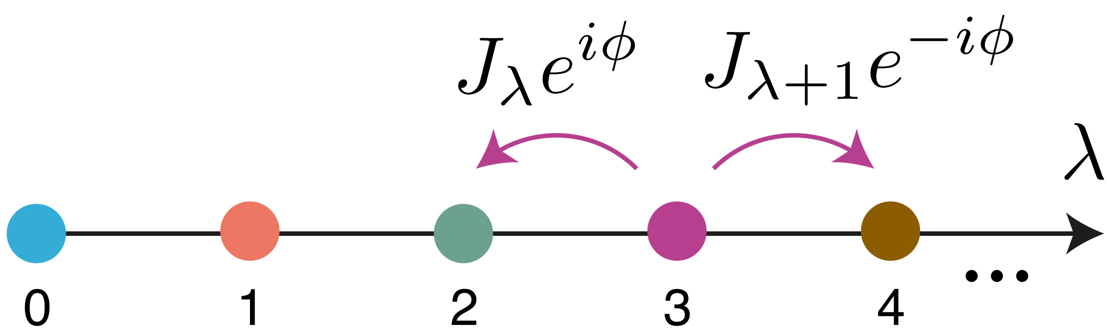
Summary

Topological physics in **four dimensions**





$$j_y = -\frac{q^3}{h^2} E_z B_{xw} \nu_2^n$$



Synthetic dimensions for cold atoms or photons

Review: *"Topological Photonics"*
Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, Iacopo Carusotto
arXiv:1802.04173

In collaboration with:

Tomoki Ozawa
(Riken, Japan)

Iacopo Carusotto
(Trento)

Ioannis Petrides
(Zurich)

Oded Zilberberg
(Zurich)

Grazia Salerno
(Brussels)

Nathan Goldman
(Brussels)

Michael Lohse
(Munich)

Christian Schweizer
(Munich)

Immanuel Bloch
(Munich)